

Discrete-Time Output-Feedback Robust Repetitive Control for a Class of Nonlinear Systems by Additive State Decomposition

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Abstract

The discrete-time robust repetitive control (RC, or repetitive controller, also designated RC) problem for nonlinear systems is both challenging and practical. This paper proposes a discrete-time output-feedback RC design for a class of systems subject to measurable nonlinearities to track reference robustly with respect to the period variation. The design relies on additive state decomposition, by which the output-feedback RC problem is decomposed into an *output-feedback* RC problem for a *linear* time-invariant system and a *state-feedback* stabilization problem for a *nonlinear* system. Thanks to the decomposition, existing controller design methods in both the frequency domain and time domain can be employed to make the robustness and discretization for a nonlinear system tractable. To demonstrate the effectiveness, an illustrative example is given.

Index Terms

Repetitive control, nonlinear systems, additive state decomposition, output feedback, uncertainties.

I. INTRODUCTION

Repetitive Control (RC, or repetitive controller, also designated RC) is a control method used specifically in tracking or rejecting periodic signals. In the past two decades, RC for linear time-invariant (LTI) systems has reached maturity. There has been little research, however, on RC for nonlinear systems [1]. This is the initial motivation of this paper. One of the major drawbacks of RC is that the control accuracy is sensitive to period variation of the external signals. It has been shown in [2] that, with a period variation as small as 1.5% for an LTI system, the gain of the internal model part of the RC drops from ∞ to 10. As a result, the tracking accuracy may be far from satisfactory, especially for high-precision control. For

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such a purpose, higher-order RCs composed of several delay blocks in series were proposed to improve the robustness of the control accuracy against period variation [2]-[5]. However, these methods cannot be applied to nonlinear systems directly as they are all based on transfer functions and frequency-domain analysis. This is the second motivation of this paper. Although controllers are often implemented by digital processors nowadays, it is rare to see a discrete-time RC design for a continuous nonlinear system. Unlike LTI systems, nonlinear systems cannot be represented as explicit, exact discrete-time models. Only approximate controller design methods can be applied by taking the discretization error as an external disturbance. This often requires the resulting closed-loop system to be input-to-state stable (ISS, or input-to-state stability, also designated ISS) with respect to the discretization error [7]. However, a linear RC system is a neutral type system in a critical case [6]. The characteristic equation of the neutral type system has an infinite sequence of roots with negative real parts approaching zero. Consequently, the ISS property cannot be obtained. Therefore, in theory, a discrete-time RC cannot be obtained by discretizing a continuous RC directly for nonlinear systems. This is the third motivation. In fact, the discretization of systems in turn brings in uncertainties in the period. Since the number of delay blocks is an integer, T/T_s has to be rounded to the nearest integer N , where T is the period of the reference signal and T_s is the sample time. So, we in fact take NT_s as the known period rather than T . This period variation is caused by discretization.

Based on the discussion above, a discrete-time robust RC problem for nonlinear systems is both challenging and practical. In this paper, we will focus on a discrete-time *output-feedback* robust RC problem for a class of systems with measurable nonlinearities. For this problem, we design a discrete-time output-feedback robust RC under a new tracking framework, named the *additive-state-decomposition-based tracking control framework* [8]. The key idea is to decompose the output-feedback RC problem into two well-solved control problems by additive state decomposition¹: an output-feedback RC for an LTI system and a state feedback stabilized control for a nonlinear system. Since the RC problem is only limited to an LTI system, existing robust higher-order RC methods can be applied directly. Moreover, according to the properties of the two control problems, we can adopt two different ways to design discrete-time controllers, i.e., discrete-time model design for the LTI component and emulation design for the nonlinear component [7]. Finally, one can combine the discrete-time output-feedback robust RC with

¹Additive state decomposition [8] is different from the lower-order subsystem decomposition methods existing in the literature. Concretely, taking the system $\dot{x}(t) = f(t, x)$, $x \in \mathbb{R}^n$ for example, it is decomposed into two subsystems: $\dot{x}_1(t) = f_1(t, x_1, x_2)$ and $\dot{x}_2(t) = f_2(t, x_1, x_2)$, where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$, respectively. The lower-order subsystem decomposition satisfies $n = n_1 + n_2$ and $x = x_1 \oplus x_2$. By contrast, the proposed additive state decomposition satisfies $n = n_1 = n_2$ and $x = x_1 + x_2$.

the discrete-time state-feedback stabilized controller to achieve the original control goal.

This paper is an extension of our previous paper [8] and focuses on the RC problem. Here we propose for a class of nonlinear systems a detailed RC design. The contributions of this paper are: i) the discrete-time RC problem is solved for a class of nonlinear systems *for the first time* (covering the discrete-time output-feedback robust RC problem for a class of continuous-time nonlinear systems); ii) more importantly, a bridge is built between existing RC design methods for LTI systems and a class of nonlinear systems so that more RC problems for nonlinear systems become tractable.

We use the following notation. \mathbb{R}^n is Euclidean space of dimension n . $\|\cdot\|$ denotes the Euclidean vector norm or induced matrix norm. The symbol $f \in \mathcal{L}_\infty$ implies that $\|f\|_\infty \triangleq \sup_{t \in [0, \infty)} \|f(t)\| < \infty$. \mathcal{L} and \mathcal{L}^{-1} denote Laplace transform and inverse Laplace transform, respectively. \mathcal{Z} and \mathcal{Z}^{-1} denote Z-transform and inverse Z-transform, respectively. \mathbb{N} denotes nonnegative integers. The following definitions can also be found in [9]. A continuous function $\alpha : [0, a) \rightarrow [0, \infty)$ is said to belong to class \mathcal{K} if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class \mathcal{K}_∞ if $a = \infty$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$. A continuous function $\beta : [0, a) \times [0, \infty) \rightarrow [0, \infty)$ is said to belong to class \mathcal{KL} if, for each fixed s , the mapping $\beta(r, s)$ belongs to \mathcal{K} with respect to r and, for each fixed r , the mapping $\beta(r, s)$ is decreasing with respect with s and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$.

II. PROBLEM FORMULATION

Consider a class of single-input-single-output (SISO) nonlinear systems [10]:

$$\begin{aligned} \dot{x} &= Ax + bu + \phi(y) + d, x(0) = x_0 \\ y &= c^T x \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ is a known stable constant matrix (see *Remark 1*), $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$ are known constant vectors, $\phi : \mathbb{R} \rightarrow \mathbb{R}^n$ is a known nonlinear function vector, $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}$ is the output, $u \in \mathbb{R}$ is the control, and $d \in \mathbb{R}^n$ is an unknown periodic bounded signal with period $T > 0$. The reference $r(t) \in \mathbb{R}$ is known and sufficiently smooth with period T . It is assumed that only $y(t)$ is available from measurements. In this paper, we consider the continuous-time system (1) by using a discrete-time controller with a sampling period $T_s > 0$, where $T = NT_s$, $N \in \mathbb{N}$. More precisely, u in (1) is constant during a sampling interval, so that $u(t) = u(kT_s) =: u(k)$, $t \in [kT_s, (k+1)T_s)$, $k \in \mathbb{N}$. In practice, T is not known exactly or is varying, namely period T is uncertain. On the other hand, we take NT_s instead of T as the period in the discrete-time controller design. Since $NT_s \neq T$ in general, T can be also considered as a variation of NT_s .

Assumption 1. The pair (A, c) is observable.

Under *Assumption 1*, the objective is to design a discrete-time output-feedback RC for the nonlinear system (1) such that $y - r$ is uniformly ultimately bounded with the ultimate bound being robust with respect to the period variation ².

Remark 1. Under *Assumption 1*, there always exists a vector $p \in \mathbb{R}^n$ such that $A + pc^T$ is stable, whose eigenvalues can be assigned freely. As a result, (1) can be rewritten as $\dot{x} = (A + pc^T)x + bu + (\phi(y) - py) + d$. Therefore, without loss of generality, we assume A to be stable.

Remark 2. The nonlinear function vector ϕ can be arbitrary. Here, we do not specify its form. Moreover, the nonlinear system (1) is allowed to be a non-minimum phase system [8].

Before proceeding further, the following preliminary result on ISS is required.

Definition 1 [7]. The system

$$\dot{x} = f(x, u(x), d_c) \quad (2)$$

is ISS with respect to d_c if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that the solutions of the system satisfy $\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|d_c\|_\infty)$, $\forall x(0), d_c \in \mathcal{L}_\infty, \forall t \geq 0$.

Suppose that the feedback is implemented by a sampler and zero-order hold as

$$u(t) = u(x(k)), t \in [kT_s, (k+1)T_s), k \in \mathbb{N}. \quad (3)$$

Then, we have

Theorem 1 [7]. If the continuous-time system (2) is ISS, then there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that given any triple of strictly positive numbers $(\Delta_x, \Delta_{d_c}, \nu)$, there exists $T^* > 0$ such that for all $T_s \in (0, T^*)$, $\|x(0)\| \leq \Delta_x, \|d_c\|_\infty \leq \Delta_{d_c}$, the solutions of the sampled-data system $\dot{x} = f(x, u(x(k)), d_c)$ satisfy:

$$\|x(k)\| \leq \beta(\|x(0)\|, kT_s) + \gamma(\|d_c\|_\infty) + \nu, k \in \mathbb{N}. \quad (4)$$

Remark 3. *Theorem 1* states that if the continuous-time closed-loop system is ISS, then the sampled-data system with the emulated controller will be semiglobally practically ISS with a sufficiently small T_s .

²Let $T = NT_s + \Delta$ be the true period, where $\Delta > 0$ is the perturbation. By using NT_s in the design, $y - r$ is uniformly ultimately bounded with the ultimate bound $d_{e\Delta} > 0$. Here robustness can be roughly understood to mean that $\frac{d_{e\Delta}}{|\Delta|}$ is small.

III. DISCRETE-TIME OUTPUT-FEEDBACK ROBUST RC BY ADDITIVE STATE DECOMPOSITION

A. Additive State Decomposition

In order to make the paper self-contained, the additive state decomposition of (1) in [8] is recalled here briefly. Consider the system (1) as the original system. We choose the primary system as follows:

$$\begin{aligned}\dot{x}_p &= Ax_p + bu_p + \phi(r) + d \\ y_p &= c^T x_p, x_p(0) = x_0.\end{aligned}\tag{5}$$

Then the secondary system is determined by the original system (1) and the primary system (5) as

$$\begin{aligned}\dot{x}_s &= Ax_s + bu_s + \phi(y) - \phi(r) \\ y_s &= c^T x_s, x_s(0) = 0\end{aligned}\tag{6}$$

where $u_s = u - u_p$. According to the additive state decomposition, we have

$$x = x_p + x_s \text{ and } y = y_p + y_s.\tag{7}$$

The secondary system (6) is further written as

$$\begin{aligned}\dot{x}_s &= Ax_s + bu_s + \phi(r + y_s + e_p) - \phi(r) \\ y_s &= c^T x_s, x_s(0) = 0\end{aligned}\tag{8}$$

where $e_p \triangleq y_p - r$. If $e_p \equiv 0$, then $(x_s, u_s) = 0$ is an equilibrium point of (8).

Controller design for the decomposed systems (5) and (6) will use their outputs or states as feedback. However, they are unknown. For such a purpose, an observer is proposed to estimate y_p and x_s .

Theorem 2 [8]. Suppose that an observer is designed to estimate y_p and x_s in (5)-(6) as follows:

$$\hat{y}_p = y - c^T \hat{x}_s\tag{9}$$

$$\dot{\hat{x}}_s = A\hat{x}_s + bu_s + \phi(y) - \phi(r), \hat{x}_s(0) = 0.\tag{10}$$

Then $\hat{y}_p \equiv y_p$ and $\hat{x}_s \equiv x_s$.

Remark 4. Additive state decomposition brings in two benefits. First, since output of the primary system and state of the secondary system can be observed, the original tracking problem for the system (1) is correspondingly decomposed into two problems: an *output-feedback* tracking problem for an LTI ‘primary’ system ($y_p \rightarrow r$) and a *state-feedback* stabilization problem for the complementary ‘secondary’ system ($x_s \rightarrow 0$). As a result, we have $y \rightarrow r$ according to (7). Since the tracking task is only assigned to the LTI system, it is therefore much easier than that for the nonlinear system (1). The state-feedback

stabilization is also easier than the output-feedback stabilization as the non-minimum phase problem is avoided. Secondly, for the two decomposed components, different discrete-time controller design methods can be employed (shown in Fig. 1) because for LTI systems an explicit, exact discrete-time model can be obtained whereas for nonlinear systems it cannot. The ISS property cannot be obtained for a traditional RC system. This implies that a sufficiently small uncertainty may cause instability. So, it is appropriate to follow the discrete-time model design for the discrete-time RC design of the linear primary system. On the other hand, a state-feedback stabilization problem for the secondary system is independent of the internal model of RCs. The resultant closed-loop system can be rendered ISS. So, we can adopt the emulation design for the discrete-time controller design of the nonlinear secondary system.

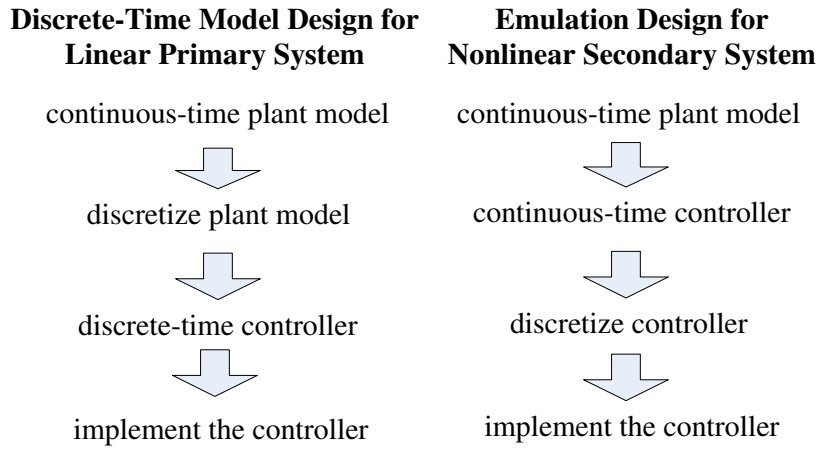


Fig. 1. Different discrete-time controller design for the primary system (5) and the secondary system (8)

B. Controller Design for Primary System and Secondary System

So far, we have decomposed the system into two systems each with its own task. In this section, we investigate controller design for each.

Problem 1. For (5), design a discrete-time output-feedback RC as

$$u_p(k) = \mathcal{Z}^{-1}(G(z)e_p(z)) \quad (11)$$

such that $e_p(k) = r(k) - y_p(k) \rightarrow \mathcal{B}(\delta)$ ³ as $k \rightarrow \infty$, where $\delta = \delta(r, d) > 0$ depends on the reference r and disturbance d .

Since (5) is an LTI system, it can be written as

$$y_p(s) = P(s)u_p(s) + d_r(s) \quad (12)$$

³ $\mathcal{B}(\delta) \triangleq \{\xi \in \mathbb{R} \mid \|\xi\| \leq \delta\}$, $\delta > 0$; the notation $x(k) \rightarrow \mathcal{B}(\delta)$ means $\min_{y \in \mathcal{B}(\delta)} |x(k) - y| \rightarrow 0$ as $k \rightarrow \infty$.

where $P(s) = c^T (sI - A)^{-1} b$ and $d_r(s) = c^T (sI - A)^{-1} \mathcal{L}(\phi(r(t)) + d(t))$. Then by using zero-order hold on the input with the sampling period T_s , the continuous-time LTI system (12) is discretized exactly as

$$y_p(z) = P(z) u_p(z) + d_r(z) \quad (13)$$

where $P(z) = c^T (zI - F)^{-1} Hb$, $F = e^{AT_s}$, $H = \int_0^{T_s} e^{As} ds$. Similar to [4],[5], we design a discrete-time output-feedback RC in the form of (11) as

$$u_p(z) = \left(1 + L(z) \frac{Q(z) W(z) z^{-N}}{1 - Q(z) W(z) z^{-N}} \right) e_p(z) \quad (14)$$

where $W(z)$ is the gain adjusting or the higher-order RC function, given by

$$W(z) = \sum_{i=1}^p w_i z^{-(i-1)N} \quad (15)$$

with $\sum_{i=1}^p w_i = 1$. For a traditional RC, $W(z) = 1$. The stability of the closed-loop system corresponding to (13) and (14) is given by *Theorem 3*.

Theorem 3. Let u_p in (13) be designed as in (14). Suppose i) $\frac{1}{1+P(z)}$, $P(z)$, $L(z)$, $Q(z)$ are stable, ii)

$$|Q(z) W(z) z^{-N} (1 - T(z) L(z))| < 1, \forall |z| = 1 \quad (16)$$

where $T(z) = \frac{P(z)}{1+P(z)}$. Then the tracking error e_p is uniformly ultimately bounded. Furthermore, if $\mathcal{Z}^{-1}((1 - Q(z) z^{-N})(r(z) - d_r(z))) \rightarrow 0$, then $e_p(k) = r(k) - y_p(k) \rightarrow 0$ as $k \rightarrow \infty$.

Proof. By substituting (14) into (13), the tracking error of the primary system can be written as

$$e_p(z) = \frac{1}{1 + P(z)} \frac{(1 - Q(z) W(z) z^{-N})(r(z) - d_r(z))}{1 - Q(z) W(z) z^{-N} (1 - T(z) L(z))} \quad (17)$$

where $T(z) = \frac{P(z)}{1+P(z)}$. A sufficient criterion for stability of the closed-loop system now becomes that $\frac{1}{1+P(z)}$ and $\frac{1 - Q(z) W(z) z^{-N}}{1 - Q(z) W(z) z^{-N} (1 - T(z) L(z))}$ are both stable. The transfer function $\frac{1}{1+P(z)}$ is stable by condition i). For stability of $\frac{1 - Q(z) W(z) z^{-N}}{1 - Q(z) W(z) z^{-N} (1 - T(z) L(z))}$, to apply the small gain theorem, $Q(z) W(z) z^{-N} (1 - T(z) L(z))$ is required to be stable first. This requires that $\frac{1}{1+P(z)}$, $P(z)$ and $Q(z)$ are stable, which are satisfied by given conditions. Therefore, if (16) holds, then $\frac{1 - Q(z) W(z) z^{-N}}{1 - Q(z) W(z) z^{-N} (1 - T(z) L(z))}$ is stable by the small gain theorem. Then the tracking error e_p is uniformly ultimately bounded. Furthermore, taking $(1 - Q(z) W(z) z^{-N})(r(z) - d_r(z))$ as a new input, since $\frac{1}{1+P(z)} \frac{1}{1 - Q(z) W(z) z^{-N} (1 - T(z) L(z))}$ is stable, $e_p(k) = r(k) - y_p(k) \rightarrow 0$ if

$$\mathcal{Z}^{-1}((1 - Q(z) z^{-N})(r(z) - d_r(z))) \rightarrow 0$$

as $k \rightarrow \infty$. \square

From *Theorem 3*, one can see that the stability depends on three main elements of the controller (14): $L(z)$, $Q(z)$ and $W(z)$. The ideal design is to let

$$1 - T(z) L(z) = 0, Q(z) = 1. \quad (18)$$

As a result, the condition (16) is satisfied and (17) becomes

$$e_p(z) = \frac{1}{1 + P(z)} (1 - W(z) z^{-N}) (r(z) - d_r(z)).$$

As a result, $e_p(k) = r(k) - y_p(k) \rightarrow 0$ with $\mathcal{Z}^{-1}((1 - W(z) z^{-N}) (r(z) - d_r(z))) \rightarrow 0$ as $k \rightarrow \infty$. However, (18) often cannot be satisfied. In the following, we will discuss how to design $L(z)$, $Q(z)$ and $W(z)$ in practice.

Remark 5 (on design of $L(z)$). In practice, the transfer function $T(z)$ might be non-minimum phase, or its relative degree is nonzero. So, we cannot find an $L(z)$ to satisfy $T(z) L(z) = 1$ exactly. Alternatively, the filter $L(z)$ can be designed by using the zero phase error tracking controller (ZPETC) algorithm as proposed in [11]. When there are zeros outside the unit circle for $T(z)$, one can rewrite $T(z)$ in the following form

$$T(z) = \frac{z^{-n_T} T_n^+(z^{-1})}{T_d(z^{-1})} = \frac{z^{-n_T} T_n^+(z^{-1}) T_n^-(z^{-1})}{T_d(z^{-1})}$$

where $T_n^+(z^{-1})$ is the cancelable part containing only the stable zeros, $T_n^-(z^{-1})$ is the uncancelable part containing only the unstable zeros, and n_T is the difference between the order of the numerator and that of the denominator. Based on the decomposition, the filter $L(z)$ is designed as

$$L(z) = \frac{T_d(z^{-1}) T_n^-(z)}{(T_n^-(1))^2 T_n^+(z^{-1})} \quad (19)$$

where $T_n^-(z)$ is obtained by replacing each z^{-1} in $T_n^-(z^{-1})$ by z . Such a design assures that the phase of $T(e^{i\omega T_s}) L(e^{i\omega T_s})$ is 0 for all frequencies ω , and the gain is 1 for low frequencies.

Remark 6 (on design of $Q(z)$). The design of $L(z)$ has assured that $T(z) L(z) \approx 1$ in the low frequency band so that the stability criterion (16) holds in the low frequency band. However, the stability criterion may be violated in the high frequency band. Based on the choice of $L(z)$, the filter $Q(z)$ is chosen to be a low-pass filter which aims to attenuate the term $|Q(z) W(z) z^{-N} (1 - T(z) L(z))|$ in the high frequency band. On the other hand, by (17), the term $1 - Q(z) W(z) z^{-N}$ will determine the tracking performance directly. It is well known that the low frequency band is often dominant in the signal $r(z) - d_r(z)$. So, an appropriate filter $Q(z)$ will make $|(1 - Q(z) W(z) z^{-N}) (r(z) - d_r(z))| \approx 0$. In practice, the trade-off between the stability and tracking performance must be taken into consideration to seek a balance.

Remark 7 (on design of $W(z)$). Gains at the harmonics are expected to be infinite [2], so $\sum_{i=1}^p w_i = 1$. With the redundant freedom, we can design appropriate weighting coefficients w_1, w_2, \dots, w_p to improve the robustness of the tracking accuracy with respect to the period variation of $r - d_r$.

So far, we have designed a discrete-time output-feedback robust RC for *Problem 1*. In the following we consider the design of a discrete-time controller for the nonlinear system (8).

Problem 2. For (8), design a controller

$$u_s(k) = u_s(x_s(k)) \quad (20)$$

such that the closed loop system is ISS with respect to the input $e_p(k)$, namely

$$\|x_s(k)\| \leq \gamma(\|e_p(k)\|) + \nu, k \in \mathbb{N} \quad (21)$$

where γ is a class \mathcal{K} function and $\nu > 0$ can be made small by reducing the sampling period T_s .

For the secondary system (8), we design a locally Lipschitz static state feedback

$$u_s(t) = u_s(x_s(t)). \quad (22)$$

Then substituting it into (8) yields

$$\dot{x}_s = f(t, x_s, e_p) \quad (23)$$

where $f(t, x_s, e_p) = Ax_s + bu_s(x_s(t)) + \phi(r(t) + c^T x_s + e_p(t)) - \phi(r(t))$. With respect to the ISS problem for (23), we have the following theorem.

Theorem 4. Suppose that there exists a continuously differentiable function $V : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \alpha_1(\|x_s\|) &\leq V(t, x_s) \leq \alpha_2(\|x_s\|) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_s} f(t, x_s, e_p) &\leq -W(x_s), \forall \|x_s\| \geq \rho(\|e_p\|) > 0 \end{aligned}$$

$\forall (t, x_s, e_p) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}$, where α_1, α_2 are class \mathcal{K}_∞ functions, ρ is a class \mathcal{K} function, and $W(x)$ is a continuous positive definite function on \mathbb{R}^n . Then, given any triple of strictly positive numbers $(\Delta_x, \Delta_{d_c}, \nu)$, there exists $T^* > 0$ such that for all $T_s \in (0, T^*)$, $\|x_s(0)\| \leq \Delta_x$, $\|e_p\|_\infty \leq \Delta_{d_c}$, the solutions of the sampled-data system (8), (20) satisfy (21), where $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Proof. We can imitate the proof of *Theorem 4.19* in [9, p. 176] to show that the continuous-time system (23) is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$. Then, based on *Theorem 1*, we can conclude that the solutions of the sampled-data system (8), (20) are semiglobally practically ISS like (4). Notice that the term $\beta(\|x_s(0)\|, kT_s)$ does not appear in (21), as $x_s(0) = 0$ implies $\beta(\|x_s(0)\|, kT_s) \equiv 0$, where $\beta \in \mathcal{KL}$. \square

Remark 8. By *Theorem 4*, if the continuous-time closed loop system is ISS by (22), then there exists a sampling period $T_s > 0$ such that the resulting closed-loop system by (20) is semiglobally practically ISS. It is difficult to give an exact T^* to ensure that *Theorem 4* is satisfied. Even if T^* is given for a general case, it will be conservative. In practice, the sampling period can be determined by simulation and experiment case-by-case.

C. Controller Integration

With the two designed controllers (14) and (20) for the two subsystems, we can combine them together to solve the original problem. The result is stated in *Theorem 5*.

Theorem 5. Suppose i) *Problems 1-2* are solved; ii) the observer-controller for system (1) is designed as:

$$\begin{aligned}\hat{y}_p(k) &= y(k) - c^T \hat{x}_s(k) \\ \hat{x}_s(k+1) &= F \hat{x}_s(k) + H b u_s(k) + H \int_0^{T_s} e^{As} (y(s) - r(s)) ds, \hat{x}_s(0) = 0\end{aligned}\quad (24)$$

and

$$\begin{aligned}u_p(k) &= \mathcal{Z}^{-1} \left(\left(1 + L(z) \frac{Q(z) W(z) z^{-N}}{1 - Q(z) W(z) z^{-N}} \right) (r(z) - \hat{y}_p(z)) \right) \\ u_s(k) &= u_s(\hat{x}_s(k)) \\ u(k) &= u_p(k) + u_s(k).\end{aligned}\quad (25)$$

Then the output of system (1) satisfies that $y(k) - r(k) \rightarrow \mathcal{B}(\delta + \|c\| \gamma(\delta) + \|c\| \nu)$ as $k \rightarrow \infty$.

Proof. By *Theorem 2*, the estimates in the observer (24) satisfy $\hat{x}_p \equiv x_p$ and $\hat{x}_s \equiv x_s$. Then the controller u_p in (25) can drive $e_p(k) = y_p(k) - r(k) \rightarrow \mathcal{B}(\delta)$ as $k \rightarrow \infty$ thanks to *Problem 1* being solved. In the following, we will further show that the controller u_s in (25) can drive $y_s(k) \rightarrow \mathcal{B}(\|c\| \gamma(\delta) + \|c\| \nu)$ as $k \rightarrow \infty$. We can conclude this proof as $u = u_p + u_s$ and $y = y_p + y_s$. *Proof of $y_s(k) \rightarrow \mathcal{B}(\|c\| \gamma(\delta) + \|c\| \nu)$.* Suppose *Problem 2* is solved. According to (21), we have $\|y_s(k)\| \leq \|c\| \|x_s(k)\| \leq \|c\| \gamma(\|e_p(k)\|) + \|c\| \nu, k \in \mathbb{N}$. Based on the result of i), we get $e_p \rightarrow \mathcal{B}(\delta)$ as $k \rightarrow \infty$. This implies that $\|e_p(k)\| \leq \delta + \varepsilon$ when $k \geq N_0$. Then $\|y_s(k)\| \leq \|c\| \beta(\|x_s(N_0)\|, k - N_0) + \|c\| \gamma(\delta + \varepsilon) + \|c\| \nu, k \geq N_0$. Since $\|c\| \beta(\|x_s(N_0)\|, k - N_0) \rightarrow 0$ as $k \rightarrow \infty$ and ε can be chosen arbitrarily small, we can conclude $y_s(k) \rightarrow \mathcal{B}(\|c\| \gamma(\delta) + \|c\| \nu)$ as $k \rightarrow \infty$. \square

Remark 9. Since the sensor sampling rate is often faster than the control rate, y is assumed to be measured continuously for the sake of simplicity so that $\hat{x}_p \equiv x_p$ and $\hat{x}_s \equiv x_s$. In (24), the term $\int_0^{T_s} e^{As} (y(s) - r(s)) ds$ can be approximated more accurately by using a finer sensor sampling rate.

IV. AN ILLUSTRATIVE EXAMPLE

A. Problem Formulation

In this paper, a single-link robot arm with a revolute elastic joint rotating in a vertical plane is served as an application [10]:

$$\begin{aligned}\dot{x} &= A_0 x + bu + \phi_0(y) + d, x(0) = x_0 \\ y &= c^T x.\end{aligned}\tag{26}$$

Here

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K}{J_l} & -\frac{F_l}{J_l} & \frac{K}{J_l} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K}{J_m} & 0 & -\frac{K}{J_m} & -\frac{F_m}{J_m} \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \phi_0(y) = \begin{bmatrix} 0 \\ -\frac{Mgl}{J_l} \sin y \\ 0 \\ 0 \end{bmatrix}, d = \begin{bmatrix} 0 \\ d_1 \\ 0 \\ d_2 \end{bmatrix}\tag{27}$$

where $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ are the link displacement (rad), link velocity (rad/s), rotor displacement (rad) and rotor velocity (rad/s), respectively; d_1 and d_2 are unknown disturbances. The initial value is assumed to be $x(0) = \begin{bmatrix} 0.05 & 0 & 0.05 & 0 \end{bmatrix}^T$. Let the link inertia $J_l = 2 \text{ kg}\cdot\text{m}^2$, motor rotor inertia $J_m = 0.5 \text{ kg}\cdot\text{m}^2$, elastic constant $k = 0.05 \text{ kg}\cdot\text{m}^2/\text{s}$, link mass $M = 0.5 \text{ kg}$, gravitational acceleration $g = 9.8 \text{ m/s}^2$, the center of mass $l = 0.5 \text{ m}$ and viscous friction coefficients $F_l = F_m = 0.2 \text{ kg}\cdot\text{m}^2/\text{s}$. The control τ is the torque delivered by the motor. The control problem here is: assuming only y is measured, design u so that y tracks a smooth enough reference r asymptotically or with a good tracking accuracy. It is easy to verify that the pair (A_0, c) is observable. It is found that A_0 is unstable. Choose $p = \begin{bmatrix} -2.10 & -1.295 & -9.36 & 3.044 \end{bmatrix}^T$. Then the system (26) can be formulated as (1) with $A = A_0 + pc^T$ and $\phi(y) = \phi_0(y) - py$, where A is stable. Assume that the desired trajectory is $r(t) = 0.05 \sin(\frac{2\pi}{T}t) + 0.1$, while the periodic disturbances are $d_1(t) = 0.04 \sin(\frac{2\pi}{T}t)$ and $d_2(t) = 0.02 \cos(\frac{2\pi}{T}t) \sin(\frac{2\pi}{T}t)$, where $T = \frac{20\pi}{3} \text{ s}$. Let the sampling period be $T_s = 0.1 \text{ s}$. Then $N = 209$.

B. Controller design

1) *Controller Design for Primary System:* Under the sampling period $T_s = 0.1 \text{ s}$, the discrete-time transfer function $P(z)$ in (13) is

$$P(z) = \frac{9.9 \times 10^{-8}(z + 9.399)(z + 0.9493)(z + 0.09589)}{(z - 0.9512)(z - 0.9418)(z - 0.9324)(z - 0.9231)}\tag{28}$$

where there exists an unstable zero -9.399 . Therefore, $P(z)$ is non-minimum phase. So, the condition (16) cannot be satisfied. We further can obtain $T(z)$. According to $T(z)$ and the ZPETC algorithm (19), we have $L(z)$. Then

$$T(z)L(z) = z^{-1} \frac{(1 + 9.399z^{-1})(1 + 9.399z)}{(1 + 9.399)^2}.$$

Choose $Q(z)$ to be an FIR filter as

$$Q(z) = 0.5 + 0.2z^{-1} + 0.2z^{-2} + 0.1z^{-3}. \quad (29)$$

According to [4], we choose

$$W(z) = 2 - z^{-N} \quad (30)$$

to improve robustness against the period variation. It is easy to check that the closed-loop system is stable. Furthermore, to compare the robustness of the tracking accuracy against the period variation, the amplitude of the transfer function in (17) with both $W(z) = 1$ and $W(z) = 2 - z^{-N}$ are plotted in Fig. 2. As shown, although the periodic components will be attenuated with $W(z) = 1$ more strongly, the higher-order RC with $W(z) = 2 - z^{-N}$ is less sensitive to the period variation in the low frequency band. Since the low frequency band is often dominant in the signal $r(z) - d_r(z)$, the higher-order RC can improve the robustness of the tracking accuracy against the period variation. This will be confirmed next.

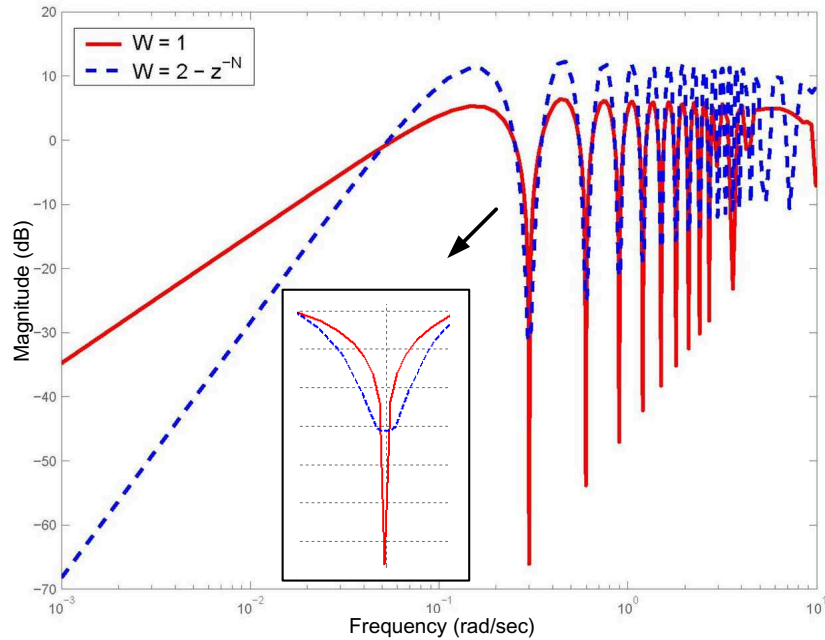


Fig. 2. The amplitude of $\frac{1}{1+P(z)} \frac{1-Q(z)W(z)z^{-N}}{1-Q(z)W(z)z^{-N}(1-T(z)L(z))}$

C. Controller Design for Secondary System

For the system (6), by the backstepping technique [9], we design

$$u_s(x_s) = \mu_1 + \frac{J_l}{K}(v + \mu_2) \quad (31)$$

where

$$\begin{aligned} v &= -7.5x_{s,1} - 19x_{s,2} - 17\eta_3 - 7\eta_4 \\ \mu_1 &= -\eta_3 + \frac{K}{J_m}x_{s,1} - \frac{K}{J_m}x_{s,3} - \frac{F_m}{J_m}x_{s,4} \\ \mu_2 &= \frac{F_l}{J_l}\eta_4 + \frac{Mgl}{J_l}(\eta_3 + \ddot{r})\cos(x_{s,1} + r) - \frac{Mgl}{J_l}[(x_{s,2} + \dot{r})^2\sin(x_{s,1} + r) + \ddot{r}\cos(r) - \dot{r}^2\sin(r)] \\ \eta_3 &= -\frac{F_l}{J_l}x_{s,2} - \frac{K}{J_l}(x_{s,1} - x_{s,3}) - \frac{Mgl}{J_l}[\sin(x_{s,1} + r) - \sin(r)] \\ \eta_4 &= -\frac{F_l}{J_l}\eta_3 - \frac{K}{J_l}(x_{s,2} - x_{s,4}) - \frac{Mgl}{J_l}[(x_{s,2} + \dot{r})\cos(x_{s,1} + r) - \dot{r}\cos(r)] \\ x_s &= \begin{bmatrix} x_{s,1} & x_{s,2} & x_{s,3} & x_{s,4} \end{bmatrix}^T. \end{aligned}$$

The controller (31) can solve *Problem 2*. (The design and proof are omitted for lake of space).

D. Controller Integration and Simulation

The final controller is given by (25), where $L(z), Q(z)$ in u_p are chosen as in *Section IV.B*, while u_s is chosen as in (31). The variables y_p and x_s are estimated by the observer (24) with the sensor sampling rate $T_{ss} = 0.01s$. In the controller combination, the variables y_p and x_s will be replaced with \hat{y}_p and \hat{x}_s . To compare the robustness of the tracking accuracy against the period variation, both $W(z) = 1$ and $W(z) = 2 - z^{-N}$ are taken into consideration, and the true period is assumed to be $\frac{20\pi}{3}(1 + \alpha)$, where α is the perturbation. The tracking error is uniformly ultimately bounded. In Fig. 3, the ultimate bound is plotted as a function of the perturbation α . As shown, the ultimate bound is small if α is small. This implies that the proposed discrete-time RC can drive y to track r . More importantly, the ultimate bound of the steady-state tracking error produced by the proposed higher-order RC with $W(z) = 2 - z^{-N}$ is less sensitive to the perturbation α in comparison with that by the traditional RC with $W(z) = 1$. Therefore, we have achieved our initial goal that a discrete-time output-feedback RC is designed for the nonlinear system (1) such that y can track r robustly with respect to the period variation.

V. CONCLUSIONS

In this paper, a discrete-time output-feedback robust RC problem for a class of systems with measurable nonlinearities was solved under the additive-state-decomposition-based tracking control framework. To

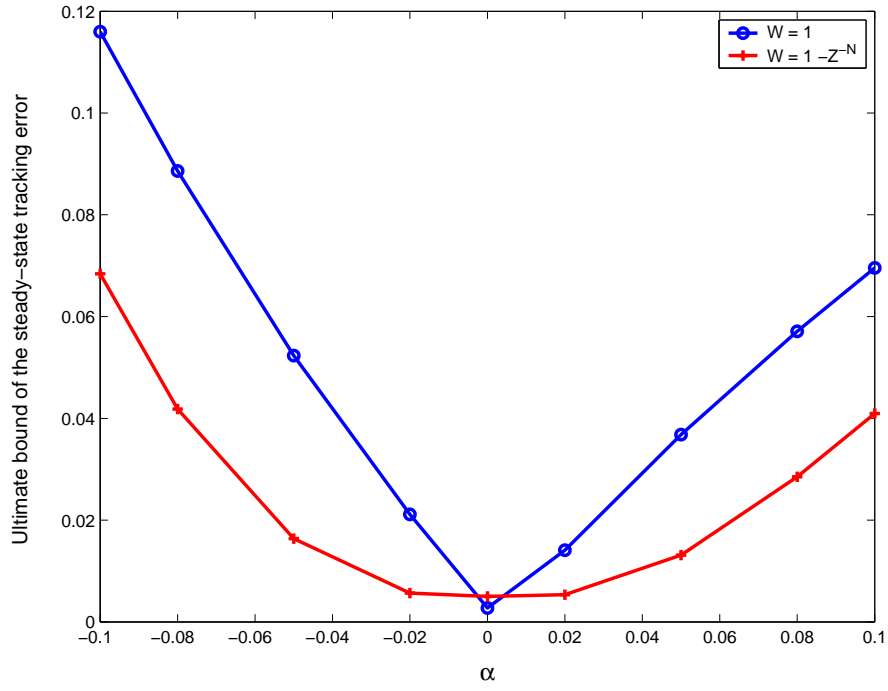


Fig. 3. The ultimate bound as function of the period-mismatch with traditional RC and higher-order RC

demonstrate its effectiveness, the design method was applied to a single-link robot arm with a revolute elastic joint rotating in a vertical plane. From the analysis and simulation, the resulting discrete-time RC can track the periodic reference and compensate for the periodic disturbance robustly with respect to the period variation. While the higher-order RC and backstepping technique are not new, our contribution is how to use additive state decomposition to solve a new and challenging RC problem, namely the discrete-time output-feedback robust RC problem for a class of nonlinear systems. Furthermore, we bridge RC design in the frequency domain for LTI systems to include a class of nonlinear systems; thus RC problems for certain nonlinear systems become tractable.

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